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On the edge-connectivity and restricted edge-connectivity of a product of graphs

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Abstract

The product graph $G_{\mathrm{m}} * G_{\mathrm{p}}$ of two given graphs G_{m} and G_{p} was defined by Bermond et al. [Large graphs with given degree and diameter II, J. Combin. Theory Ser. B 36 (1984) 32–48]. For this kind of graphs we provide bounds for two connectivity parameters (λ and λ', edge-connectivity and restricted edge-connectivity, respectively), and state sufficient conditions to guarantee optimal values of these parameters. Moreover, we compare our results with other previous related ones for permutation graphs and cartesian product graphs, obtaining several extensions and improvements. In this regard, for any two connected graphs G_{m}, G_{p} of minimum degrees $\delta(G_{\mathrm{m}})$, $\delta(G_{\mathrm{p}})$, respectively, we show that $\lambda(G_{\mathrm{m}} * G_{\mathrm{p}})$ is lower bounded by both $\delta(G_{\mathrm{m}}) + \lambda(G_{\mathrm{p}})$ and $\delta(G_{\mathrm{p}}) + \lambda(G_{\mathrm{m}})$, an improvement of what is known for the edge-connectivity of $G_{\mathrm{m}} \times G_{\mathrm{p}}$.

Keywords: Edge-connectivity; Restricted edge-connectivity; Permutation graphs; Cartesian product

1. Introduction

A usual objective in network design is the extension of a given interconnection system to a larger and fault-tolerant one so that the communication delay among nodes of the new network is small enough. One interesting model for this kind of extension is a *permutation graph*—introduced by Chartrand and Harary in [9]—which is simply obtained by taking two disjoint copies of a given graph G and adding a perfect matching between the two copies. The diameter of a permutation graph has been investigated in [14], whereas [3,13,16–18] are some examples of references where the study of the connectivity of permutation graphs has been addressed. One can naturally wonder if a graph defined similarly from a larger number of copies of G (connecting them somehow) can be still seen as a useful model for the extension of a network under the requirements of small diameter and large connectivity. Such these graphs were introduced a pair of decades ago by Bermond et al. (see [4,5], for example). In this regard, the *product graph* $G_{\mathrm{m}} * G_{\mathrm{p}}$ of two given graphs G_{m}, G_{p} (defined in [4]) can be considered as a natural generalization of a permutation graph, as will be noticed in Section 2.

This work deals with such product graphs $G_{\mathrm{m}} * G_{\mathrm{p}}$, for which we provide bounds for two connectivity parameters (λ and λ', edge-connectivity and restricted edge-connectivity, respectively). Moreover, we present sufficient conditions

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to guarantee optimal values of these parameters for product graphs. Our results will be compared with other previous related ones: results for the connectivity of permutation graphs and of cartesian product graphs. Before proceeding, it seems useful to devote Section 2 to recall some basic definitions and to set the notation that will be used in the rest of the paper. In Section 3 we present our results on edge-connectivity and restricted edge-connectivity of a product graph $G_{\mathrm{m}} * G_{\mathrm{p}}$, and prove them in Section 4.

2. Terminology and notation

We follow [10] for graph-theoretical terminology and notation not defined here. A graph $G=(V,E)$ always means a *simple graph* (without loops and multiple edges), where $V=V(G)$ is the vertex set and $E=E(G)$ is the edge set. The *degree* of a vertex v is denoted by $d(v)=d_G(v)$, whereas $\delta=\delta(G)$ and $\Delta=\Delta(G)$ stand for the *minimum degree* and the *maximum degree* of G, respectively. For every $u\in V$, the *edge-neighborhood of* u is $\omega(u)=\omega_G(u)=\{e\in E : e \text{ is incident with } u\}$. For every $uv\in E$, the *edge-boundary of* uv, denoted by $\omega(uv)=\omega_G(uv)$, is the set of edges $\omega(uv)=(\omega(u)\cup\omega(v))-uv$, and $|\omega(uv)|=d(u)+d(v)-2$ is called the *edge-degree of* uv. If $E\neq\emptyset$, then $\xi=\xi(G)$ denotes the *minimum edge-degree* of G; that is, $\xi(G)=\min\{d(u)+d(v)-2 : uv\in E\}$. Observe that $2\delta(G)-2\leqslant\xi(G)\leqslant\delta(G)+\Delta(G)-2$. Two distinct edges $xy, uv\in E(G)$ are called *independent* if $\{x,y\}\cap\{u,v\}=\emptyset$. A *matching* is a set of edges that are pairwise independent. A *perfect matching* between two disjoint graphs G_1, G_2 with the same order n is a matching consisting of n edges such that each of them has one endvertex in G_1 and the other one in G_2. The *diameter* of G is written as $D=D(G)$, which is finite if G is *connected*.

An *edge-cut* of a connected graph G is a set S of edges such that $G-S$ is not connected. An edge-cut is called *minimal* if it does not contain any other edge-cut. An edge-cut is called *minimum* if no other edge-cut with fewer edges exists; observe that every minimum edge-cut is also minimal. The *edge-connectivity* of G, denoted by $\lambda=\lambda(G)$, is the cardinality of a minimum edge-cut, and it is widely known that $\lambda(G)\leqslant\delta(G)$. Even though no edge-cuts exist for K_1, the equality $\lambda(K_1)=0$ is taken. A connected graph G is called *maximally edge-connected* if $\lambda(G)=\delta(G)$. In this paper we are also interested in the so-called *restricted edge-connectivity* $\lambda'=\lambda'(G)$, a parameter that was introduced by Esfahanian and Hakimi [12] as follows:

$$\lambda'=\min\{|X| : X\subset E \text{ is a restricted edge-cut}\},$$

where an edge-cut $X\subset E$ is called *restricted* if no vertex u of the graph is such that $\omega(u)\subset X$. A restricted edge-cut S of G is called *minimum* if no other restricted edge-cut with fewer edges exists, hence $|S|=\lambda'(G)$. It is readily seen that every minimum restricted edge-cut is a minimal edge-cut. As proved in [12], $\lambda'(G)$ exists when the connected graph G is not a star and has at least four vertices, in which case $\lambda(G)\leqslant\lambda'(G)\leqslant\xi(G)$ holds. When $\lambda'(G)$ exists, G is said to be λ'*-connected*, and G is called λ'*-optimal* in case $\lambda'(G)=\xi(G)$. This parameter λ' can be used to study how connected a graph is in a more accurate manner than only by means of λ. In fact, $\lambda'(G)>\delta(G)$ is equivalent to saying that the graph G is *edge-superconnected*, i.e., to saying that every minimum edge-cut is equal to the edge-neighborhood of some vertex of minimum degree, see [6,7]. Some sufficient conditions for a graph to be λ'-optimal have been given in terms of the girth in [1,2].

The construction of new graphs from two given ones is not unusual at all. In this regard, Chartrand and Harary introduced in [9] the concept of *permutation graph* as follows. For a graph G and a permutation π of $V(G)$, the permutation graph $G^{(\pi)}$ is defined by taking two disjoint copies of G and adding a perfect matching joining each vertex v in the first copy to $\pi(v)$ in the second copy. Examples of these graphs include hypercubes, prisms and some generalized Petersen graphs.

Later, Bermond et al. [5] introduced the concept of *compound graph* $G[\Gamma]$ on the graphs Γ and G. One similar type of compound graph is the *product graph* of two given graphs, defined in [4] by Bermond et al. in the following way.

Definition 1 (*Bermond et al. [4]*). Let $G_{\mathrm{m}}=(V(G_{\mathrm{m}}),E(G_{\mathrm{m}}))$ and $G_{\mathrm{p}}=(V(G_{\mathrm{p}}),E(G_{\mathrm{p}}))$ be two graphs. Let us give an arbitrary orientation to the edges of G_{m}, in such a way that an arc from vertex x to vertex y is denoted by e_{xy}. For each arc e_{xy}, let $\pi_{e_{xy}}$ be a permutation of $V(G_{\mathrm{p}})$. Then the product graph $G_{\mathrm{m}} * G_{\mathrm{p}}$ has $V(G_{\mathrm{m}})\times V(G_{\mathrm{p}})$ as vertex set, two vertices (x,x'), (y,y') being adjacent iff either

$$x=y \quad \text{and} \quad x'y'\in E(G_{\mathrm{p}})$$

or

$$e_{xy} \text{ is an arc} \quad \text{and} \quad y' = \pi_{e_{xy}}(x').$$

The product graph $G_m * G_p$ can be viewed as formed by $|V(G_m)|$ disjoint copies of G_p, each arc e_{xy} indicating that some perfect matching between the copies G_p^x, G_p^y (generated by the vertices x and y of G_m, respectively) is added. So the graph G_m is usually called the *main graph* and G_p is called the *pattern graph* of the product graph $G_m * G_p$. Moreover, every edge of $G_m * G_p$ that belongs to any of the $|E(G_m)|$ perfect matchings between copies of G_p is an *intercopy edge* of $G_m * G_p$.

Observe that if we choose $\pi_{e_{xy}}(x') = x'$ for any arc e_{xy} then $G_m * G_p = G_m \times G_p$. Furthermore, if G_m is K_2 we have $K_2 * G = G^{(\pi)}$, a permutation graph. Hence, $G_m * G_p$ can be considered as a *generalized permutation graph*.

Some relations between the minimum degree, the maximum degree, and the diameter of a product graph $G_m * G_p$ with the corresponding parameters of its main graph and its pattern graph can be found in [4].

Lemma 2 (*Bermond et al. [4]*). *Let G_m and G_p be two graphs. Then, for every product graph $G_m * G_p$*:

(i) $\delta(G_m * G_p) = \delta(G_m) + \delta(G_p)$, $\Delta(G_m * G_p) = \Delta(G_m) + \Delta(G_p)$.
(ii) *If both G_m and G_p are connected, then $G_m * G_p$ is also connected and*

$$D(G_m) \leqslant D(G_m * G_p) \leqslant D(G_m) + D(G_p).$$

3. Results

The following proposition is a key point for the rest of our results. In order to present it, some appropriate notation follows (this notation will also be used very often in Section 4). If $W \subset E(G)$ is an edge-cut of a product graph $G = G_m * G_p$ and G_p^x is any given copy in G of the pattern graph, we will say that G_p^x *is split by* W if both $V(H) \cap V(G_p^x) \neq \emptyset$ and $V(H^*) \cap V(G_p^x) \neq \emptyset$ hold for some two components H, H^* of $G - W$.

Proposition 3. *Let G_m and G_p be two connected graphs, $|V(G_m)| \geqslant 2$, $|V(G_p)| \geqslant 2$. Let $W \subset E(G)$ be a minimal edge-cut of $G = G_m * G_p$, and let r denote the number of copies in G of G_p that are split by W, $0 \leqslant r \leqslant |V(G_m)|$. The following statements hold*:

(i) $|W| \geqslant \lambda(G_m)|V(G_p)|$ *if* $r = 0$; $|W| \geqslant (\delta(G_m) + 1)\lambda(G_p)$ *if* $r \geqslant \delta(G_m) + 1$.
(ii) $|W| \geqslant r(\delta(G_m) + \delta(G_p) - r + 1) \geqslant \delta(G_m) + \delta(G_p)$ *if* $1 \leqslant r \leqslant \delta(G_m)$.

For any two connected graphs G_m and G_p, the inequality $\lambda(G_m * G_p) \leqslant \delta(G_m) + \delta(G_p)$ holds as a consequence of Lemma 2. When $|V(G_m)| \geqslant 2$ and $|V(G_p)| \geqslant 2$, Proposition 3 allows us to derive a lower bound for $\lambda(G_m * G_p)$, after considering any minimum edge-cut $W \subset E(G_m * G_p)$ of $G_m * G_p$. Moreover, if $|V(G_m)| = 1$ or $|V(G_p)| = 1$ it is clear that $\min\{\lambda(G_m)|V(G_p)|, (\delta(G_m)+1)\lambda(G_p), \delta(G_m) + \delta(G_p)\} = 0$. Joining together these facts we can write the following theorem.

Theorem 4. *For every two connected graphs G_m and G_p*:

$$\min\{\lambda(G_m)|V(G_p)|, (\delta(G_m) + 1)\lambda(G_p), \delta(G_m) + \delta(G_p)\} \leqslant \lambda(G_m * G_p) \leqslant \delta(G_m) + \delta(G_p).$$

It was proved in [11] that $\lambda(G_m \times G_p) \geqslant \lambda(G_m) + \lambda(G_p)$. Now Theorem 4 allows us to obtain an improvement of this result.

Corollary 5. *For every two connected graphs G_m and G_p*:

$$\lambda(G_m * G_p) \geqslant \min\{\delta(G_m) + \lambda(G_p), \delta(G_p) + \lambda(G_m)\}.$$

Further, $G_m * G_p$ is maximally edge-connected when both G_m and G_p are maximally edge-connected.

Another consequence of Theorem 4 is the following corollary, which can be also obtained from the results in [23].

Corollary 6. *Let $G \neq K_1$ be a connected graph, and let $k \geqslant 0$ be an integer. Then*

$$\lambda(K_{k+1} \times G) = \min\{(k+1)\lambda(G), k + \delta(G)\}.$$

Now, the result in [16,18] for the edge-connectivity of a permutation graph $G^{(\pi)}$ is a direct consequence of Theorem 4 and Corollary 6, when recalling that $G^{(\pi)}$ can be written as $K_2 * G$ and that $G^{(\mathrm{id})}$ stands for the cartesian product $K_2 \times G$ (id is the identity permutation).

Corollary 7 (*Lai [16], Piazza and Ringeisen [18]*). *Let G be a connected graph. Then, for every permutation π of $V(G)$*:

$$\min\{2\lambda(G), \delta(G) + 1\} = \lambda(G^{(\mathrm{id})}) \leqslant \lambda(G^{(\pi)}) \leqslant \delta(G) + 1.$$

We can also derive at this point the following important result.

Theorem 8. *Let G_{m} and G_{p} be two connected graphs. Then the graph $G_{\mathrm{m}} * G_{\mathrm{p}}$ is maximally edge-connected if any of the following conditions hold*:

(i) $|V(G_{\mathrm{p}})| \geqslant |V(G_{\mathrm{m}})| \geqslant 2$, $-2 \leqslant \delta(G_{\mathrm{m}}) - \delta(G_{\mathrm{p}}) \leqslant 2$, *and* $\lambda(G_{\mathrm{p}}) \geqslant 2$.
(ii) $|V(G_{\mathrm{m}})| = |V(G_{\mathrm{p}})|$ *and* $\delta(G_{\mathrm{m}}) = \delta(G_{\mathrm{p}})$.

Corollary 9. *For every connected graph G, the graph $G * G$ is maximally edge-connected.*

One may ask whether maximal edge-connectivity can be guaranteed or not for product graphs from earlier known results on the edge-connectivity of a graph. To shed some light on this question, some of the best known sufficient conditions for maximal edge-connectivity of a connected graph are brought together (in chronological order) in the following theorem.

Theorem 10. *Let G be a connected graph of order n, maximum and minimum degrees Δ and δ respectively, diameter D and girth g. Then G is maximally edge-connected if any of the following assertions hold*:

(i) $\delta \geqslant \lfloor n/2 \rfloor$ [8].
(ii) $D \leqslant 2$ [19].
(iii) $n > (\delta - 1)(\Delta^{D-1} + \Delta^2 - 2)/(\Delta - 1)$ [15].
(iv) $\delta \geqslant 3$ *and* $n > (\delta - 1)((\Delta - 1)^{D-1} + \Delta - 3)/(\Delta - 2) + \Delta - 1$ [21].
(v) $D \leqslant \begin{cases} g-1 & \textit{if } g \textit{ is odd;} \\ g-2 & \textit{if } g \textit{ is even.} \end{cases}$ [21].

Let us compare the results in Theorem 10 with Corollary 9, for any connected graph G with $\delta(G) \geqslant 2$. It is easy to see that the condition in point (i) of Theorem 10 does not hold for any $G * G$ such that $G \neq K_3$, and also that $G * G$ does not satisfy the condition in point (ii) of that theorem if $G \neq K_n$. Hence, the usefulness of Corollary 9 is clear with respect to these first two points of Theorem 10. The question is not so simple for the remaining points of Theorem 10, and its applicability to a product graph $G * G$ depends on the actual graph G and/or the set of perfect matchings between copies of G. For example, consider $G = C_n$, the cycle of length $n \geqslant 3$, see Fig. 1. We have $\delta(C_n * C_n) = \Delta(C_n * C_n) = 4$, $\lfloor n/2 \rfloor \leqslant D(C_n * C_n) \leqslant 2\lfloor n/2 \rfloor$, $g(C_n * C_n) \leqslant n$, and $|V(C_n * C_n)| = n^2$. Hence, the right-hand side of the inequality in point (iii) of Theorem 10 is not less than

$$4^{\lfloor n/2 \rfloor - 1} + 14.$$

But, as $4^{\lfloor n/2 \rfloor - 1} + 14 \geqslant n^2$ easily holds for every $n \geqslant 8$, $n \neq 9$, it turns out that the above referred point (iii) cannot be used to deduce that $\lambda(C_n * C_n) = \delta(C_n * C_n)$ for these values of the integer n, as Corollary 9 guarantees. Similarly, $\lambda(C_n * C_n) = \delta(C_n * C_n)$ follows for every $n \geqslant 3$ from Corollary 9 but does not follow from point (iv) of Theorem 10 when $n \geqslant 10$. Finally, one can see after some calculations (which compute the maximum possible girth for a 3-regular subgraph of $C_n * C_n$ consisting of two cycles C_n and a perfect matching between them) that the inequalities in point

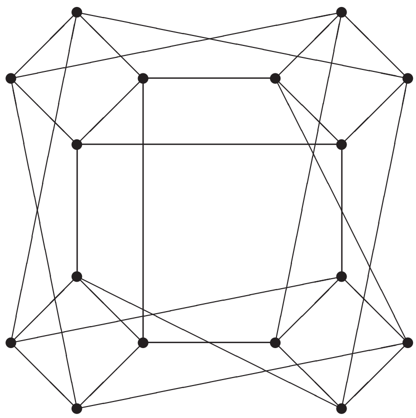

Fig. 1. A graph $C_4 * C_4$.

(v) of Theorem 10 do not hold when $n \geqslant 14$, so the usefulness of Corollary 9 for the edge-connectivity of $C_n * C_n$ is again exhibited.

Before approaching the study of the restricted edge-connectivity of a product graph $G_m * G_p$, we next present a simple lemma which relates its minimum edge-degree with some parameters of its main graph G_m and its pattern graph G_p.

Lemma 11. *Let G_m and G_p be two graphs, both containing some edge. Then, for every product graph $G_m * G_p$:*

$$2\delta(G_m) + 2\delta(G_p) - 2 \leqslant \xi(G_m * G_p) \leqslant \min\{\xi(G_p) + 2\delta(G_m), \xi(G_m) + \delta(G_p) + \Delta(G_p)\}.$$

Theorem 12. *Let $G_m \neq K_1$ and $G_p \neq K_1$ be two connected graphs. Then the graph $G = G_m * G_p$ is λ'-connected and*

$$\min\{\lambda(G_m)|V(G_p)|, (\delta(G_m) + 1)\lambda(G_p), \delta(G_m) + 2\delta(G_p) - 1\} \leqslant \lambda'(G) \leqslant \xi(G).$$

Shieh proved in [20] that any graph $G_m \times G_p$ (except $K_2 \times K_n, n \geqslant 2$) is edge-superconnected provided that both the connected graphs $G_m \neq K_1$ and $G_p \neq K_1$ are maximally edge-connected regular graphs. As an immediate consequence of Theorem 12, we obtain an improvement of this result for product graphs $G_m * G_p$ in which the regularity constraint is not necessary.

Corollary 13. *Let G_m and G_p be two maximally edge-connected graphs with $\delta(G_m) \geqslant 2$ and $\delta(G_p) \geqslant 2$. Then the graph $G_m * G_p$ is edge-superconnected.*

We next provide bounds for the restricted edge-connectivity of a product graph $G_m * G_p$, from which we will be able to give some results for the λ'-optimality of these graphs (see [22] for some related results, for the particular case of cartesian product graphs).

Theorem 14. *Let G_m and $G_p \neq K_3$ be two connected graphs. If $\delta(G_p) \geqslant \Delta(G_m) + 1 \geqslant 2$, then the graph $G = G_m * G_p$ is λ'-connected and*

$$\min\{\lambda(G_m)|V(G_p)|, (\delta(G_m) + 1)\lambda'(G_p), \delta(G_m)(\delta(G_p) + 1) + \lambda'(G_p), \xi(G)\} \leqslant \lambda'(G) \leqslant \xi(G).$$

Concerning the restricted edge-connectivity of a permutation graph $G^{(\pi)} = K_2 * G$, in [3] was proved that if G is a connected graph with $|V(G)| \geqslant \xi(G) + 2$ and $\delta(G) \geqslant 2$, then any permutation graph $G^{(\pi)}$ is λ'-connected and

$$\min\{2\lambda'(G), \lambda'(G) + \delta(G), \xi(G^{(\pi)})\} \leqslant \lambda'(G^{(\pi)}) \leqslant \xi(G^{(\pi)}).$$

Now, we obtain an improvement of this result as a direct consequence of Theorem 14.

Corollary 15. *Let G be a connected graph with $|V(G)| \geqslant \xi(G)+2$ and $\delta(G) \geqslant 2$. Then, for every permutation π of $V(G)$, the permutation graph $G^{(\pi)}$ is λ'-connected and*

$$\min\{2\lambda'(G), \lambda'(G)+\delta(G)+1, \xi(G^{(\pi)})\} \leqslant \lambda'(G^{(\pi)}) \leqslant \xi(G^{(\pi)}).$$

Corollary 16. *Let G_{m} and G_{p} be two connected graphs. If $\delta(G_{\mathrm{p}}) \geqslant \Delta(G_{\mathrm{m}})+1 \geqslant 2$, $\lambda(G_{\mathrm{m}})|V(G_{\mathrm{p}})| \geqslant \xi(G_{\mathrm{p}})+2\delta(G_{\mathrm{m}})$ and $\lambda'(G_{\mathrm{p}})+\delta(G_{\mathrm{m}}) \geqslant \xi(G_{\mathrm{p}})+2$, then the graph $G=G_{\mathrm{m}} * G_{\mathrm{p}}$ is λ'-connected and $\lambda'(G)=\xi(G)$.*

Corollary 17. *Let G be a connected graph, with $\delta(G) \geqslant 3$ and $\Delta(G) \leqslant 2\delta(G)-1$. Then, the graph $G*(G*G)$ is λ'-connected and $\lambda'(G*(G*G))=\xi(G*(G*G))$.*

4. Proofs

For the following proofs, it must be recalled that the inequality $a \cdot b \geqslant a+b-1$ holds for any pair of integers $a, b \geqslant 1$.

Proof of Proposition 3. Because W is minimal we have that $G-W$ consists of exactly two connected components, H and H^*. Observe that $|W| \geqslant r\lambda(G_{\mathrm{p}})$ holds, because at least $\lambda(G_{\mathrm{p}})$ edges must be deleted from G in order to split (by W) each of the considered r copies of G_{p}.

(i) If $r \geqslant \delta(G_{\mathrm{m}})+1$ then $|W| \geqslant (\delta(G_{\mathrm{m}})+1)\lambda(G_{\mathrm{p}})$ follows directly. Suppose now that $r=0$. Then all the edges in W are intercopy edges and correspond to $t \geqslant 1$ perfect matchings between copies of G_{p} that appear in G as a replacement of t edges of G_{m}. Moreover, the set of these t edges of G_{m} must be an edge-cut of G_{m} (for if not, $G-W$ is still connected), hence $t \geqslant \lambda(G_{\mathrm{m}})$. Thus $|W| \geqslant \lambda(G_{\mathrm{m}})|V(G_{\mathrm{p}})|$.

(ii) Let $V(G_{\mathrm{m}})=\{x_1, x_2, \ldots, x_n\}$. Without loss of generality, assume that the r split (by W) copies of G_{p} are $G_{\mathrm{p}}^{x_1}, G_{\mathrm{p}}^{x_2}, \ldots, G_{\mathrm{p}}^{x_r}$ corresponding to vertices $x_1, x_2, \ldots, x_r$ of G_{m}, $1 \leqslant r \leqslant \delta(G_{\mathrm{m}})$. For every $j=1, \ldots, r$, let us write $V(G_{\mathrm{p}}^{x_j})=V_j \cup V_j^*$, with $V_j \subset V(H)$, $V_j^* \subset V(H^*)$, and let us denote $k_j=\min\{|V_j|, |V_j^*|\}$; moreover, let us call s_j to the number of edges of $G_{\mathrm{p}}^{x_j}$ joining vertices in V_j to vertices in V_j^*. Taking into account that each x_j $(j=1, \ldots, r)$ is adjacent in G_{m} to at least $\delta(G_{\mathrm{m}})-(r-1)$ other vertices x_q of G_{m}, $q \geqslant r+1$, then from copy $G_{\mathrm{p}}^{x_j}$ we have at least as many edges in W as $k_j(\delta(G_{\mathrm{m}})-(r-1))+s_j$. Thus we obtain

$$|W| \geqslant \sum_{j=1}^{r}(k_j(\delta(G_{\mathrm{m}})-(r-1))+s_j). \tag{1}$$

Let us study the terms of the above sum according to the value of k_j.

If $k_j \geqslant \delta(G_{\mathrm{p}})$, using that $s_j \geqslant \lambda(G_{\mathrm{p}}) \geqslant 1$ we have

$$k_j(\delta(G_{\mathrm{m}})-r+1)+s_j \geqslant k_j+\delta(G_{\mathrm{m}})-r+s_j \geqslant \delta(G_{\mathrm{p}})+\delta(G_{\mathrm{m}})-r+1. \tag{2}$$

If $k_j \leqslant \delta(G_{\mathrm{p}})-1$, assuming without loss of generality that $k_j=|V_j^*|$, we can write

$$k_j(k_j-1) \geqslant \sum_{u \in V_j^*} d_{G_{\mathrm{p}}^{x_j}}(u)-s_j \geqslant k_j\delta(G_{\mathrm{p}})-s_j,$$

hence

$$s_j \geqslant k_j(\delta(G_{\mathrm{p}})-k_j+1). \tag{3}$$

Therefore, from (3) we have

$$k_j(\delta(G_{\mathrm{m}})-r+1)+s_j \geqslant k_j(\delta(G_{\mathrm{m}})+\delta(G_{\mathrm{p}})-r-k_j+2) \geqslant \delta(G_{\mathrm{m}})+\delta(G_{\mathrm{p}})-r+1, \tag{4}$$

because when $k_j \in \{1, \ldots, \delta(G_{\mathrm{p}})-1\}$ it is not difficult to see that the middle term of the above chain of inequalities takes its minimum value when $k_j=1$. Hence, from (1), (2) and (4) it follows

$$|W| \geqslant r(\delta(G_{\mathrm{m}})+\delta(G_{\mathrm{p}})-r+1). \tag{5}$$

Since the minimum value for the right-hand side of (5) is taken when $r = 1$, we have finally that

$$|W| \geqslant r(\delta(G_{\rm m}) + \delta(G_{\rm p}) - r + 1) \geqslant \delta(G_{\rm m}) + \delta(G_{\rm p}),$$

ending the proof. □

Proof of Corollary 5. The claimed inequality easily holds when $|V(G_{\rm m})| = 1$ or $|V(G_{\rm p})| = 1$, that is to say, when $\delta(G_{\rm m}) = \lambda(G_{\rm m}) = 0$ or $\delta(G_{\rm p}) = \lambda(G_{\rm p}) = 0$. Hence, suppose $\lambda(G_{\rm m}) \geqslant 1$ and $\lambda(G_{\rm p}) \geqslant 1$ from now on. Taking into account that $|V(G_{\rm p})| \geqslant \delta(G_{\rm p}) + 1$, we have

$$\begin{aligned}
&\lambda(G_{\rm m})|V(G_{\rm p})| \geqslant \lambda(G_{\rm m})(\delta(G_{\rm p}) + 1) \geqslant \lambda(G_{\rm m}) + \delta(G_{\rm p}),\\
&(\delta(G_{\rm m}) + 1)\lambda(G_{\rm p}) \geqslant \delta(G_{\rm m}) + \lambda(G_{\rm p}),\\
&\delta(G_{\rm m}) + \delta(G_{\rm p}) \geqslant \min\{\delta(G_{\rm m}) + \lambda(G_{\rm p}), \delta(G_{\rm p}) + \lambda(G_{\rm m})\}.
\end{aligned}$$

Hence the inequality follows directly from Theorem 4. Furthermore, as a consequence, $\lambda(G_{\rm m} * G_{\rm p}) = \delta(G_{\rm m}) + \delta(G_{\rm p})$ holds provided that $\lambda(G_{\rm m}) = \delta(G_{\rm m})$ and $\lambda(G_{\rm p}) = \delta(G_{\rm p})$. □

Proof of Corollary 6. When $k = 0$ it is clear that $K_{k+1} \times G = K_1 \times G = G$, hence $\lambda(K_{k+1} \times G) = \lambda(G)$ and the result follows. Then, assume that $k \geqslant 1$. We have $\lambda(K_{k+1}) = \delta(K_{k+1}) = k$ and $|V(G)| \geqslant \delta(G) + 1$, and then

$$\lambda(K_{k+1})|V(G)| \geqslant k(\delta(G) + 1) \geqslant \delta(G) + k.$$

Therefore, following Theorem 4, we can write

$$k + \delta(G) \geqslant \lambda(K_{k+1} \times G) \geqslant \min\{(k + 1)\lambda(G), k + \delta(G)\} \geqslant 2,$$

the last inequality being a consequence of $k \geqslant 1$ and $G \neq K_1$. The result clearly follows when $k + \delta(G) \leqslant (k + 1)\lambda(G)$, hence we can suppose $(k + 1)\lambda(G) \leqslant k + \delta(G)$.

To end the proof it suffices to show that there exists in $K_{k+1} \times G$ an edge-cut W such that $|W| = (k + 1)\lambda(G)$. Let $\widetilde{W} \subset E(G)$ be a minimum edge-cut of the pattern graph, $|\widetilde{W}| = \lambda(G)$. Let $V(K_{k+1}) = \{x_1, x_2, \ldots, x_{k+1}\}$, and let $W_1, W_2, \ldots, W_{k+1}$ be the corresponding copies of $\widetilde{W}$ in $G^{x_1}, G^{x_2}, \ldots, G^{x_{k+1}}$, respectively. Then, $W_1 \cup W_2 \cup \cdots \cup W_{k+1}$ is clearly an edge-cut of $K_{k+1} \times G$, of cardinality $|W_1 \cup W_2 \cup \cdots \cup W_{k+1}| = (k + 1)\lambda(G)$. □

Proof of Theorem 8. Let $\delta = \min\{\delta(G_{\rm m}), \delta(G_{\rm p})\}$. Observe that $\delta(G_{\rm m} * G_{\rm p}) = \delta(G_{\rm m}) + \delta(G_{\rm p}) \leqslant 2\delta + 2$ for both cases (i), (ii), and also that $|V(G_{\rm p})| \geqslant 1 + \delta(G_{\rm p}) \geqslant 1 + \delta$.

(i) As $\lambda(G_{\rm p}) \geqslant 2$ by hypothesis, we have

$$(\delta(G_{\rm m}) + 1)\lambda(G_{\rm p}) \geqslant 2(\delta + 1) \geqslant \delta(G_{\rm m}) + \delta(G_{\rm p}). \tag{6}$$

Now, we claim that

$$\lambda(G_{\rm m})|V(G_{\rm p})| \geqslant \delta(G_{\rm m}) + \delta(G_{\rm p}), \tag{7}$$

with $\lambda(G_{\rm m}) \geqslant 1$ because $G_{\rm m} \neq K_1$. Indeed, when $\lambda(G_{\rm m}) \geqslant 2$ we obtain

$$\lambda(G_{\rm m})|V(G_{\rm p})| \geqslant 2(1 + \delta) \geqslant \delta(G_{\rm m}) + \delta(G_{\rm p}),$$

so (7) holds. Inequality (7) also holds clearly if $\delta(G_{\rm m}) = 1$ because

$$\lambda(G_{\rm m})|V(G_{\rm p})| \geqslant |V(G_{\rm p})| \geqslant 1 + \delta(G_{\rm p}) = \delta(G_{\rm m}) + \delta(G_{\rm p}).$$

Hence, suppose that $1 = \lambda(G_{\rm m}) < \delta(G_{\rm m})$, so from item (i) of Theorem 10 we get $|V(G_{\rm m})| \geqslant 2\delta(G_{\rm m}) + 2 \geqslant 2\delta + 2$. Recalling the hypothesis $|V(G_{\rm p})| \geqslant |V(G_{\rm m})|$, we have

$$\lambda(G_{\rm m})|V(G_{\rm p})| = |V(G_{\rm p})| \geqslant 2\delta + 2 \geqslant \delta(G_{\rm m}) + \delta(G_{\rm p}),$$

and (7) is again true. Thus, from (6), (7) and from Theorem 4 it follows that $\lambda(G_{\rm m} * G_{\rm p}) = \delta(G_{\rm m}) + \delta(G_{\rm p})$.

(ii) In this case, $\delta=\delta(G_{\rm m})=\delta(G_{\rm p})$, and we want to prove that $\lambda(G_{\rm m}*G_{\rm p})=2\delta$. When $|V(G_{\rm m})|=|V(G_{\rm p})|=1$, both $G_{\rm m}$ and $G_{\rm p}$ are equal to the graph K_1, hence $G_{\rm m}*G_{\rm p}=K_1$ and the result is obvious. So $G_{\rm m}\neq K_1$ and $G_{\rm p}\neq K_1$ must be assumed. Moreover from Corollary 5 it follows $\lambda(G_{\rm m}*G_{\rm p})=2\delta$ if $\delta=1$, so assume that $\delta\geqslant 2$. Furthermore, the previous item (i) allows us to assume also that $\lambda(G_{\rm p})=1$, which in turn yields $|V(G_{\rm m})|=|V(G_{\rm p})|\geqslant 2\delta(G_{\rm p})+2=2\delta+2$ as a consequence of point (i) of Theorem 10.

Let $W\subset E(G_{\rm m}*G_{\rm p})$ be any minimum edge-cut of $G_{\rm m}*G_{\rm p}$, $|W|=\lambda(G_{\rm m}*G_{\rm p})$, and let H, H^* be the two connected components of $(G_{\rm m}*G_{\rm p})-W$. Let r denote the number of split (by W) copies of $G_{\rm p}$ in $G_{\rm m}*G_{\rm p}$. Following Proposition 3 we have $|W|\geqslant\lambda(G_{\rm m})|V(G_{\rm p})|=\lambda(G_{\rm m})|V(G_{\rm m})|>2\delta$ when $r=0$, and also $|W|\geqslant\delta(G_{\rm m})+\delta(G_{\rm p})=2\delta$ if $1\leqslant r\leqslant\delta$. Moreover, if $r\geqslant 2\delta$ then the inequality $|W|\geqslant 2\delta$ holds because clearly $|W|\geqslant r$.

Hence, suppose that $\delta+1\leqslant r\leqslant 2\delta-1$, and assume that the r considered split copies of $G_{\rm p}$ are $G_{\rm p}^{x_1}, G_{\rm p}^{x_2},\ldots,G_{\rm p}^{x_r}$ (corresponding to vertices $x_1,x_2,\ldots,x_r$ of $G_{\rm m}$). As $|V(G_{\rm m})|\geqslant 2\delta+2$ and $r\leqslant 2\delta-1$, there must exist some vertex $x_k\in V(G_{\rm m})$, $k>r$, such that x_k is adjacent to some vertex in $\{x_1,x_2,\ldots,x_r\}$, say to x_1. As in Proposition 3 we write $V(G_{\rm p}^{x_j})=V_j\cup V_j^*$, with $V_j\subset V(H)$, $V_j^*\subset V(H^*)$ (for $j=1,\ldots,r$), and denote by s_j the number of edges of $G_{\rm p}^{x_j}$ joining vertices in V_j to vertices in V_j^*. Since W contains all the edges connecting vertices in $G_{\rm p}^{x_k}$ with vertices in V_1, or W contains all the edges connecting vertices in $G_{\rm p}^{x_k}$ with vertices in V_1^*, then the set of edges W contains at least $k_1=\min\{|V_1|,|V_1^*|\}$ edges that are incident with vertices of $G_{\rm p}^{x_k}$. If $k_1\leqslant\delta-1$, then reasoning as in Proposition 3 we have that $s_1\geqslant k_1(\delta-(k_1-1))\geqslant\delta$, because the minimum is attained for $k_1=1$. Hence we obtain

$$|W|\geqslant k_1+s_1+\sum_{j=2}^{r}s_j\geqslant k_1+\delta+r-1>2\delta,$$

because $r\geqslant\delta+1$. And if $k_1\geqslant\delta$ then

$$|W|\geqslant k_1+\sum_{j=1}^{r}s_j\geqslant\delta+r\geqslant 2\delta+1>2\delta.$$

Then, we have shown that $|W|\geqslant 2\delta$ in any case, and the proof of (ii) is complete. □

Proof of Lemma 11. Observe that $\xi(G_{\rm m})$, $\xi(G_{\rm p})$ and $\xi(G_{\rm m}*G_{\rm p})$ can be defined, because $E(G_{\rm m})\neq\emptyset$, $E(G_{\rm p})\neq\emptyset$, and so $E(G_{\rm m}*G_{\rm p})\neq\emptyset$. Notice also that the lower bound for $\xi(G_{\rm m}*G_{\rm p})$ follows easily because $\xi(G_{\rm m}*G_{\rm p})\geqslant 2\delta(G_{\rm m}*G_{\rm p})-2=2\delta(G_{\rm m})+2\delta(G_{\rm p})-2$ from Lemma 2.

For the upper bound, consider first some edge $yy'\in E(G_{\rm p})$ such that $|\omega_{G_{\rm p}}(yy')|=\xi(G_{\rm p})$, and let $x\in V(G_{\rm m})$ be a vertex such that $d_{G_{\rm m}}(x)=\delta(G_{\rm m})$. Taking $u=(x,y)$ and $v=(x,y')$, it turns out that $uv\in E(G_{\rm m}*G_{\rm p})$ hence we can write

$$\xi(G_{\rm m}*G_{\rm p})\leqslant|\omega_{G_{\rm m}*G_{\rm p}}(uv)|=|\omega_{G_{\rm p}}(yy')|+2\delta(G_{\rm m})=\xi(G_{\rm p})+2\delta(G_{\rm m}).$$

Second, let $xx'\in E(G_{\rm m})$ be an edge such that $|\omega_{G_{\rm m}}(xx')|=\xi(G_{\rm m})$, and let $y\in V(G_{\rm p})$ be a vertex such that $d_{G_{\rm p}}(y)=\delta(G_{\rm p})$. Consider the intercopy edge $uv\in E(G_{\rm m}*G_{\rm p})$ with $u=(x,y)$ and $v=(x',y')$. Hence

$$\xi(G_{\rm m}*G_{\rm p})\leqslant|\omega_{G_{\rm m}*G_{\rm p}}(uv)|=|\omega_{G_{\rm m}}(xx')|+d_{G_{\rm p}}(y)+d_{G_{\rm p}}(y')\leqslant\xi(G_{\rm m})+\delta(G_{\rm p})+\Delta(G_{\rm p}),$$

and the proof is complete. □

Proof of Theorem 12. Observe that $|V(G_{\rm m})|\geqslant 2$ and $|V(G_{\rm p})|\geqslant 2$ implies $|V(G)|\geqslant 4$. Besides, G is not a star because $\delta(G)=\delta(G_{\rm m})+\delta(G_{\rm p})\geqslant 2$, hence G is λ'-connected and $\lambda'(G)\leqslant\xi(G)$.

Let $W\subset E(G)$ be any minimum restricted edge-cut and let H, H^* be the two connected components of $G-W$. Let r denote the number of split copies of $G_{\rm p}$ in $G_{\rm m}*G_{\rm p}$. Proposition 3 provides the corresponding lower bounds for $|W|$ when $r=0$ and also when $r\geqslant\delta(G_{\rm m})+1$. Again from Proposition 3 we get $|W|\geqslant r(\delta(G_{\rm m})+\delta(G_{\rm p})-r+1)$ if $2\leqslant r\leqslant\delta(G_{\rm m})$. Now the minimum of $r(\delta(G_{\rm m})+\delta(G_{\rm p})-r+1)$ is attained for $r=2$ when $2\leqslant r\leqslant\delta(G_{\rm m})$, thus we deduce that $|W|\geqslant 2(\delta(G_{\rm m})+\delta(G_{\rm p})-1)$ in this case.

Suppose that $r=1$ from now on, and assume that the considered split copy of $G_{\rm p}$ is $G_{\rm p}^{x_1}$ (corresponding to vertex x_1 of $G_{\rm m}$). Let $V(G_{\rm p}^{x_1})=V_1\cup V_1^*$ be such that $V_1\subset V(H)$ and $V_1^*\subset V(H^*)$, and let s_1 be the number of edges joining vertices in V_1 with vertices in V_1^*. Now if β_1 is the number of edges of $G_{\rm m}$ that join x_1 with vertices x_j whose copy $G_{\rm p}^{x_j}$ in G is contained in H^*, then we have

$$\lambda'(G)=|W|\geqslant|V_1|\beta_1+|V_1^*|(d_{G_{\rm m}}(x_1)-\beta_1)+s_1. \tag{8}$$

First suppose that $\beta_1=0$. This means that all copies $G_{\rm p}^{x_j}$ corresponding to neighbors x_j of x_1 in $G_{\rm m}$ are contained in H, which implies $|V_1^*|\geqslant 2$ because W is a restricted edge-cut. Moreover, if $|V_1^*|\geqslant\delta(G_{\rm p})+1$ then, taking into account that $s_1\geqslant\lambda(G_{\rm p})$, we get

$$|W|\geqslant|V_1^*|\delta(G_{\rm m})+s_1\geqslant(\delta(G_{\rm p})+1)\delta(G_{\rm m})+\lambda(G_{\rm p})>(\delta(G_{\rm m})+1)\lambda(G_{\rm p}),$$

and the result follows. So assume that $2\leqslant|V_1^*|\leqslant\delta(G_{\rm p})$. Reasoning as in Proposition 3, we have that $s_1\geqslant|V_1^*|(\delta(G_{\rm p})-|V_1^*|+1)$. Hence we obtain

$$|W|\geqslant|V_1^*|(\delta(G_{\rm m})+\delta(G_{\rm p})-|V_1^*|+1)\geqslant 2(\delta(G_{\rm m})+\delta(G_{\rm p})-1),$$

and we are done.

Second suppose that $\beta_1=d_{G_{\rm m}}(x_1)$. In this case (8) becomes $|W|\geqslant|V_1|\delta(G_{\rm m})+s_1$ and $|V_1|\geqslant 2$ because W is a restricted edge-cut. In a similar way as for $\beta_1=0$ we obtain again $|W|\geqslant 2(\delta(G_{\rm m})+\delta(G_{\rm p})-1)$, and the result holds.

Finally suppose that $1\leqslant\beta_1\leqslant d_{G_{\rm m}}(x_1)-1$. In this case, it follows from (8) that

$$\lambda'(G)=|W|\geqslant|V_1|+\beta_1+|V_1^*|+d_{G_{\rm m}}(x_1)-\beta_1-2+s_1\geqslant|V(G_{\rm p})|+\delta(G_{\rm m})+\lambda(G_{\rm p})-2.$$

Now if $|V(G_{\rm p})|\leqslant 2\delta(G_{\rm p})+1$ then $\lambda(G_{\rm p})=\delta(G_{\rm p})$ by Theorem 10 (i), and as $|V(G_{\rm p})|\geqslant\delta(G_{\rm p})+1$ we get

$$\lambda'(G)=|W|\geqslant\delta(G_{\rm m})+2\delta(G_{\rm p})-1,$$

and we have finished. If $|V(G_{\rm p})|\geqslant 2\delta(G_{\rm p})+2$ then

$$\lambda'(G)=|W|\geqslant\delta(G_{\rm m})+2\delta(G_{\rm p})+\lambda(G_{\rm p}),$$

which completes the proof. □

Proof of Theorem 14. Observe that $G_{\rm m}\neq K_1$ because $\Delta(G_{\rm m})\geqslant 1$. So $\delta(G)=\delta(G_{\rm m})+\delta(G_{\rm p})\geqslant 3$ and $|V(G)|\geqslant\delta(G)+1\geqslant 4$, hence G is λ'-connected and $\lambda'(G)\leqslant\xi(G)$. Notice also that $\delta(G_{\rm p})\geqslant 2$ and $G_{\rm p}\neq K_3$ implies that $G_{\rm p}$ is not a star and its order is at least four, hence $\lambda'(G_{\rm p})$ exists.

Let $W\subset E(G)$ be a minimum restricted edge-cut of G, $|W|=\lambda'(G)$. Thus $G-W$ has no isolated vertex and consists of exactly two connected components, H and H^*. If $V(G_{\rm m})=\{x_1,\dots,x_n\}$ we write $W=W_1\cup\cdots\cup W_n\cup W_{cc}$, where $W_j\subset E(G_{\rm p}^{x_j})$ for each $j\in\{1,\dots,n\}$, and W_{cc} is only composed by intercopy edges. Clearly, if $W_j\neq\emptyset$ then W_j is an edge-cut of $G_{\rm p}^{x_j}$ because W has minimum cardinality.

First, let us suppose that one of the components of $G-W$, say H, consists of the subgraph induced by one vertex $u=(x_j,y)\in V(G_{\rm p}^{x_j})$ plus a number of q neighbors $v_1,\dots,v_q$ in $V(G)\backslash V(G_{\rm p}^{x_j})$, $1\leqslant q\leqslant d_{G_{\rm m}}(x_j)$ (observe that $q\neq 0$, because u cannot be isolated in $G-W$), such that $d_H(v_i)=1$ for each $i\in\{1,\dots,q\}$. With this structure for H, we have

$$|W|\geqslant d_{G_{\rm p}}(y)+(d_G(v_1)-1)+\sum_{i=2}^{q}(d_G(v_i)-1)+(d_{G_{\rm m}}(x_j)-q).$$

But $d_G(v_i)\geqslant\delta(G)\geqslant 3$ for each $1\leqslant i\leqslant q\leqslant d_{G_{\mathrm{m}}}(x_j)$, hence

$$\begin{aligned}\lambda'(G)=|W|&\geqslant d_{G_{\mathrm{p}}}(y)+(d_G(v_1)-1)+(q-1)+(d_{G_{\mathrm{m}}}(x_j)-q)\\&=d_G(u)+d_G(v_1)-2=|\omega_G(uv_1)|\geqslant\xi(G),\end{aligned}$$

and the theorem holds. Then, we assume henceforth that neither H nor H^* has this structure.

Now, let us see that each nonempty $W_j\subset E(G_{\mathrm{p}}^{x_j})$ is a restricted edge-cut of $G_{\mathrm{p}}^{x_j}$. Indeed, suppose that $G_{\mathrm{p}}^{x_j}-W_j$ isolates one vertex $u\in V(G_{\mathrm{p}}^{x_j})$, and without loss of generality suppose that $u=(x_j,y)\in V(H)$. If $v_1,\dots,v_q\in V(G)\backslash V(G_{\mathrm{p}}^{x_j})$ are the neighbors of u in H, we have $d_H(v_i)\geqslant 2$ for some $i\in\{1,\dots,q\}$. Let $B(u)\subset\omega_G(u)\backslash E(G_{\mathrm{p}}^{x_j})$ be the set of edges joining in H the vertex u with any vertex $v\notin V(G_{\mathrm{p}}^{x_j})$ such that $d_H(v)\geqslant 2$. Then the set of edges of G

$$W'=(W\backslash\omega_{G_{\mathrm{p}}^{x_j}}(u))\cup B(u)$$

is also a restricted edge-cut of G with cardinality $|W'|=\lambda'(G)-d_{G_{\mathrm{p}}^{x_j}}(u)+|B(u)|\leqslant\lambda'(G)-\delta(G_{\mathrm{p}})+\Delta(G_{\mathrm{m}})$. But taking into account the hypothesis $\delta(G_{\mathrm{p}})\geqslant\Delta(G_{\mathrm{m}})+1$, it follows $|W'|\leqslant\lambda'(G)-1$, which is not possible according to the fact that W was a minimum restricted edge-cut. Therefore, each nonempty $W_j\subset E(G_{\mathrm{p}}^{x_j})$ is a restricted edge-cut of $G_{\mathrm{p}}^{x_j}$, so $|W_j|\geqslant\lambda'(G_{\mathrm{p}})$.

Let r be the number of copies in G of G_{p} that are split by W, $0\leqslant r\leqslant|V(G_{\mathrm{m}})|$. When $r\geqslant\delta(G_{\mathrm{m}})+1$, we have that $\lambda'(G)=|W|\geqslant(\delta(G_{\mathrm{m}})+1)\lambda'(G_{\mathrm{p}})$, because at least $r\cdot\lambda'(G_{\mathrm{p}})$ edges must be deleted from G in order to split (by W) the considered r copies of G_{p}. Then, the theorem follows in this case. When $r=0$, we get $\lambda'(G)=|W|\geqslant\lambda(G_{\mathrm{m}})\cdot|V(G_{\mathrm{p}})|$ from Proposition 3, and the result also holds.

Finally, the case $1\leqslant r\leqslant\delta(G_{\mathrm{m}})$ is discussed similarly as in the proof of Proposition 3 (and the same notation is adopted). If $G_{\mathrm{p}}^{x_1},G_{\mathrm{p}}^{x_2},\dots,G_{\mathrm{p}}^{x_r}$ are the considered copies of the pattern graph that are split by W, observe that $k_j=\min\{|V_j|,|V_j^*|\}\geqslant 2$ and $s_j=|W_j|\geqslant\lambda'(G_{\mathrm{p}})$ hold for every $j=1,\dots,r$, because W_j is a restricted edge-cut of $G_{\mathrm{p}}^{x_j}$. Recall that for every $j=1,\dots,r$, we have at least as many edges in W from copy $G_{\mathrm{p}}^{x_j}$ as $k_j(\delta(G_{\mathrm{m}})-(r-1))+s_j$. Hence (see expression (1) in the proof of Proposition 3)

$$|W|\geqslant\sum_{j=1}^{r}(k_j(\delta(G_{\mathrm{m}})-(r-1))+s_j).\tag{9}$$

Let us study the terms of the above sum according to the value of k_j.

If $k_j\geqslant\delta(G_{\mathrm{p}})+1$, using that $s_j\geqslant\lambda'(G_{\mathrm{p}})$ we have

$$k_j(\delta(G_{\mathrm{m}})-r+1)+s_j\geqslant(\delta(G_{\mathrm{p}})+1)(\delta(G_{\mathrm{m}})-r+1)+\lambda'(G_{\mathrm{p}}).\tag{10}$$

If $2\leqslant k_j\leqslant\delta(G_{\mathrm{p}})$, assuming without loss of generality that $k_j=|V_j^*|$, and taking an edge u_0u_1 of the subgraph induced by V_j^* we get

$$k_j(k_j-1)\geqslant d_{G_{\mathrm{p}}^{x_j}}(u_0)+d_{G_{\mathrm{p}}^{x_j}}(u_1)+\sum_{u\in V_j^*\backslash\{u_0,u_1\}}d_{G_{\mathrm{p}}^{x_j}}(u)-s_j\geqslant\xi(G_{\mathrm{p}})+2+(k_j-2)\delta(G_{\mathrm{p}})-s_j,$$

hence

$$s_j\geqslant k_j(\delta(G_{\mathrm{p}})-k_j+1)+\xi(G_{\mathrm{p}})-2\delta(G_{\mathrm{p}})+2.\tag{11}$$

Thus, from (11) it follows

$$\begin{aligned}k_j(\delta(G_{\mathrm{m}})-r+1)+s_j&\geqslant k_j(\delta(G_{\mathrm{m}})+\delta(G_{\mathrm{p}})-r-k_j+2)+\xi(G_{\mathrm{p}})-2\delta(G_{\mathrm{p}})+2\\&\geqslant 2(\delta(G_{\mathrm{m}})-r+1)+\xi(G_{\mathrm{p}}),\end{aligned}\tag{12}$$

having used for the second inequality that $k_j(\delta(G_{\mathrm{m}})+\delta(G_{\mathrm{p}})-r-k_j+2)+\xi(G_{\mathrm{p}})-2\delta(G_{\mathrm{p}})+2$ takes its minimum value for $k_j=2$ when $k_j\in\{2,\dots,\delta(G_{\mathrm{p}})\}$. Then, from (9), (10) and (12), for some integer h, $0\leqslant h\leqslant r$, it follows that

$$\begin{aligned}|W|&\geqslant h((\delta(G_{\mathrm{p}})+1)(\delta(G_{\mathrm{m}})-r+1)+\lambda'(G_{\mathrm{p}}))+(r-h)(2(\delta(G_{\mathrm{m}})-r+1)+\xi(G_{\mathrm{p}}))\\&=(2r+h(\delta(G_{\mathrm{p}})-1))(\delta(G_{\mathrm{m}})-r+1)+(r-h)\xi(G_{\mathrm{p}})+h\lambda'(G_{\mathrm{p}}).\end{aligned}\tag{13}$$

When $h = r$ we obtain from (13):

$$|W| \geqslant r((\delta(G_{\mathrm{p}}) + 1)(\delta(G_{\mathrm{m}}) - r + 1) + \lambda'(G_{\mathrm{p}})).$$

For $h = 0$, (13) yields

$$|W| \geqslant r(2(\delta(G_{\mathrm{m}}) - r + 1) + \xi(G_{\mathrm{p}})).$$

As both these lower bounds for $|W|$ take their minimum values when $r = 1$, we have

$$\lambda'(G) = |W| \geqslant \begin{cases} (\delta(G_{\mathrm{p}}) + 1)\delta(G_{\mathrm{m}}) + \lambda'(G_{\mathrm{p}}) & \text{if } h = r, \\ 2\delta(G_{\mathrm{m}}) + \xi(G_{\mathrm{p}}) \geqslant \xi(G) & \text{if } h = 0, \end{cases}$$

and the theorem holds. Hence, suppose $1 \leqslant h \leqslant r - 1$, so $2 \leqslant r \leqslant \delta(G_{\mathrm{m}})$. In this case, from (13) we get

$$|W| \geqslant (2r + \delta(G_{\mathrm{p}}) - 1)(\delta(G_{\mathrm{m}}) - r + 1) + \xi(G_{\mathrm{p}}) + \lambda'(G_{\mathrm{p}}).$$

But the right-hand term of this inequality takes its minimum value when $r = \delta(G_{\mathrm{m}})$, and so

$$\lambda'(G) = |W| \geqslant 2\delta(G_{\mathrm{m}}) + \delta(G_{\mathrm{p}}) - 1 + \xi(G_{\mathrm{p}}) + \lambda'(G_{\mathrm{p}}) > 2\delta(G_{\mathrm{m}}) + \xi(G_{\mathrm{p}}) \geqslant \xi(G),$$

completing the proof. □

Proof of Corollary 16. From Lemma 11 it follows that $\xi(G_{\mathrm{p}}) + 2\delta(G_{\mathrm{m}}) \geqslant \xi(G)$. Moreover, both hypotheses $\lambda'(G_{\mathrm{p}}) + \delta(G_{\mathrm{m}}) \geqslant \xi(G_{\mathrm{p}}) + 2$ and $\delta(G_{\mathrm{p}}) \geqslant \Delta(G_{\mathrm{m}}) + 1$ imply that $\lambda'(G_{\mathrm{p}}) \geqslant \delta(G_{\mathrm{p}}) + 1$. Thus combining Theorem 14 and Lemma 11 we obtain

$$\begin{aligned} \xi(G) \geqslant \lambda'(G) &\geqslant \min\{(\delta(G_{\mathrm{m}}) + 1)\lambda'(G_{\mathrm{p}}), \delta(G_{\mathrm{m}})(\delta(G_{\mathrm{p}}) + 1) + \lambda'(G_{\mathrm{p}}), \xi(G)\} \\ &= \min\{\delta(G_{\mathrm{m}})(\delta(G_{\mathrm{p}}) + 1) + \lambda'(G_{\mathrm{p}}), \xi(G)\} \\ &\geqslant \min\{2\delta(G_{\mathrm{m}}) + \delta(G_{\mathrm{m}}) + \lambda'(G_{\mathrm{p}}), \xi(G)\} = \xi(G). \end{aligned}$$

Hence $\lambda'(G) = \xi(G)$. □

Proof of Corollary 17. First notice that $G * (G * G)$ is λ'-connected, because by hypotheses

$$\delta(G * G) = 2\delta(G) \geqslant \Delta(G) + 1 \geqslant 4. \tag{14}$$

Lemma 2 allows us to write

$$\begin{aligned} \xi(G * G) + 2\delta(G) &\leqslant \delta(G * G) + \Delta(G * G) - 2 + 2\delta(G) \\ &= 4\delta(G) + 2\Delta(G) - 2 \leqslant 6\Delta(G) - 2, \end{aligned} \tag{15}$$

$$\begin{aligned} \xi(G * (G * G)) &\leqslant \delta(G * (G * G)) + \Delta(G * (G * G)) - 2 \\ &= 3\delta(G) + 3\Delta(G) - 2 \leqslant 9\delta(G) - 5. \end{aligned} \tag{16}$$

Now, from (15) and taking into account that $\lambda(G) \geqslant 1$ and $\Delta(G) \geqslant 3$ we obtain

$$\lambda(G)|V(G * G)| = \lambda(G)|V(G)|^2 \geqslant (\Delta(G) + 1)^2 \geqslant 6\Delta(G) - 2 \geqslant \xi(G * G) + 2\delta(G). \tag{17}$$

Moreover, $G * G$ is maximally edge-connected because of Corollary 9, so $\lambda'(G * G) \geqslant \lambda(G * G) = 2\delta(G)$. Hence, taking into account (16) and $\delta(G) \geqslant 3$ we have

$$\begin{aligned} &(\delta(G) + 1)\lambda'(G * G) \geqslant (\delta(G) + 1) \cdot 2\delta(G) \geqslant 9\delta(G) - 5 \geqslant \xi(G * (G * G)), \\ &\delta(G)\delta(G * G) + \lambda'(G * G) \geqslant (\delta(G) + 1) \cdot 2\delta(G) \geqslant \xi(G * (G * G)). \end{aligned} \tag{18}$$

Therefore taking $G_{\mathrm{m}} = G$ and $G_{\mathrm{p}} = G * G$ in Theorem 14, from (14), (17) and (18) we have finally that $\lambda'(G * (G * G)) = \xi(G * (G * G))$. □

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